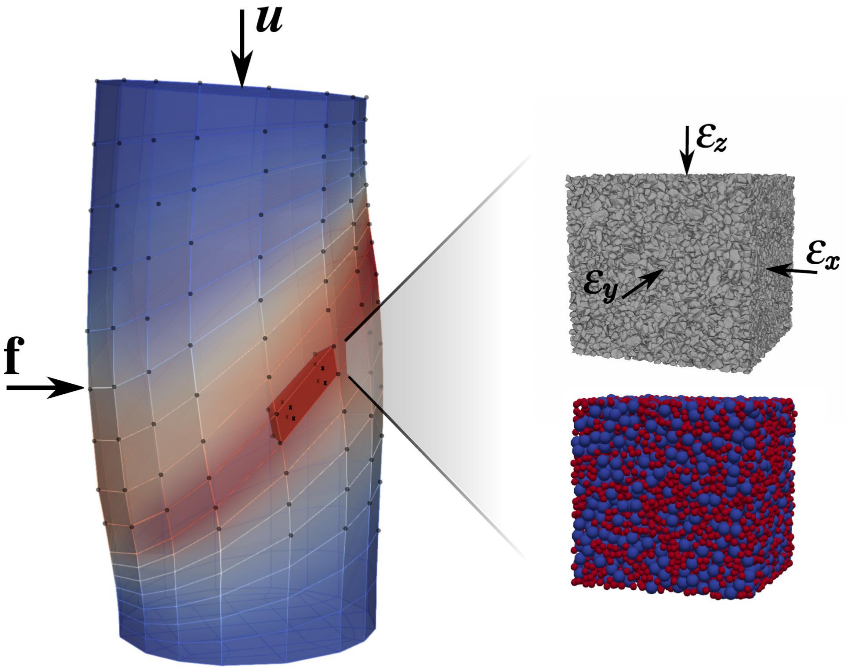


CIVIL-408 Multiscale Modeling in Mechanics

Prof. Kostas Karapiperis

Exercises - Week 4

Review of linear elasticity

Linear elasticity

$$\sigma_{ij} = \mathbb{C}_{ijkl}\varepsilon_{kl}$$ where $\mathbb{C}_{ijkl}$ is a 4th-order tensor with 81 coefficients.

- Symmetry of stress tensor $\sigma_{ij} = \sigma_{ji} \rightarrow \mathbb{C}_{ijkl} = \mathbb{C}_{jikl}$
- Symmetry of strain tensor $\varepsilon_{kl} = \varepsilon_{lk} \rightarrow \mathbb{C}_{ijkl} = \mathbb{C}_{ijlk}$
- Strain energy energy $1/2\,\varepsilon_{ij}\mathbb{C}_{ijkl}\varepsilon_{kl} \rightarrow \mathbb{C}_{ijkl} = \mathbb{C}_{klij}$

reduction to 21 coefficients

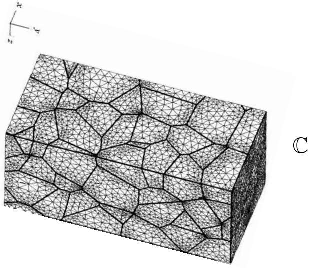
Voigt notation

$$\underbrace{\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{pmatrix}}_{\sigma_\alpha} = \underbrace{\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & \text{sym.} & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{pmatrix}}_{\mathbb{C}_{\alpha\beta}} \underbrace{\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\\ 2\varepsilon_{12}\end{pmatrix}}_{\varepsilon_\beta}$$

Review of material symmetry

Anisotropic

$$\mathbb{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym.} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{pmatrix}$$



21 parameters

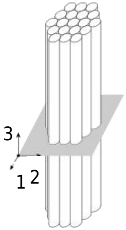
Orthotropic

$$\mathbb{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym.} & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix}$$

9 parameters

Transversally isotropic

$$\mathbb{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym.} & & & C_{44} & 0 \\ & & & & & (C_{11}-C_{12})/2 \end{pmatrix}$$



5 parameters

Isotropic

$$\mathbb{C} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym.} & & & C_{44} & 0 \\ & & & & & C_{44} \end{pmatrix}$$

$$C_{44} = (C_{11} - C_{12})/2$$

2 parameters (Lamé moduli)

$$\lambda := \frac{E\nu}{(1-2\nu)(1+\nu)} \qquad \mu := \frac{E}{2(1+\nu)}$$

Material symmetry

What type of material is this?

Material 1

1. Isotropic
2. Transversally isotropic
3. Orthotropic
4. Anisotropic

$$
\mathbb{C} = \begin{pmatrix} a & c & d & 0 & 0 & 0 \\ & a & d & 0 & 0 & 0 \\ & & b & 0 & 0 & 0 \\ & & & e & 0 & 0 \\ & \text{sym.} & & & e & 0 \\ & & & & & a/2 \end{pmatrix}
$$

$$a, b, c, d, e \in \mathbb{R}^+$$

Material 2

1. Isotropic
2. Transversally isotropic
3. Orthotropic
4. Anisotropic

Isotropic layers

λ_1, μ_1

λ_2, μ_2

...

Material symmetry

What type of material is this?

- **Material 1**

1. Isotropic
2. Transversally isotropic
3. Orthotropic
4. Anisotropic

$$\mathbb{C} = \begin{pmatrix} a & c & d & 0 & 0 & 0 \\ & a & d & 0 & 0 & 0 \\ & & b & 0 & 0 & 0 \\ & & & e & 0 & 0 \\ & \text{sym.} & & & e & 0 \\ & & & & & a/2 \end{pmatrix}$$

$$a, b, c, d, e \in \mathbb{R}^+$$

- **Material 2**

1. Isotropic
2. Transversally isotropic
3. Orthotropic
4. Anisotropic

Isotropic layers

λ_1, μ_1

λ_2, μ_2

...

Isotropic layered media: Analytical solution

JOURNAL OF GEOPHYSICAL RESEARCH VOLUME 67, No. 11 OCTOBER 1962

Long-Wave Elastic Anisotropy Produced by Horizontal Layering

GEORGE E. BACKUS

Institute of Geophysics and Department of Earth Sciences
University of California, La Jolla

Abstract. A horizontally layered inhomogeneous medium, isotropic or transversely isotropic, is considered, whose properties are constant or nearly so when averaged over some vertical height l'. For waves longer than l' the medium is shown to behave like a homogeneous, or nearly homogeneous, transversely isotropic medium whose density is the average density and whose elastic coefficients are algebraic combinations of averages of algebraic combinations of the elastic coefficients of the original medium. The nearly homogeneous medium is said to be 'long-wave equivalent' to the original medium. Conditions on the five elastic coefficients of a homogeneous transversely isotropic medium are derived which are necessary and sufficient for the medium to be 'long-wave equivalent' to a horizontally layered isotropic medium. Further conditions are also derived which are necessary and sufficient for the homogeneous medium to be 'long-wave equivalent' to a horizontally layered isotropic medium consisting of only two different homogeneous isotropic materials. Except in singular cases, if the latter two-layered medium exists at all, its proportions and elastic coefficients are uniquely determined by the elastic coefficients of the homogeneous transversely isotropic medium. The observed variations in crustal P-wave velocity with depth, obtained from well logs, are shown to be large enough to explain some of the observed crustal anisotropies as due to layering of isotropic material.

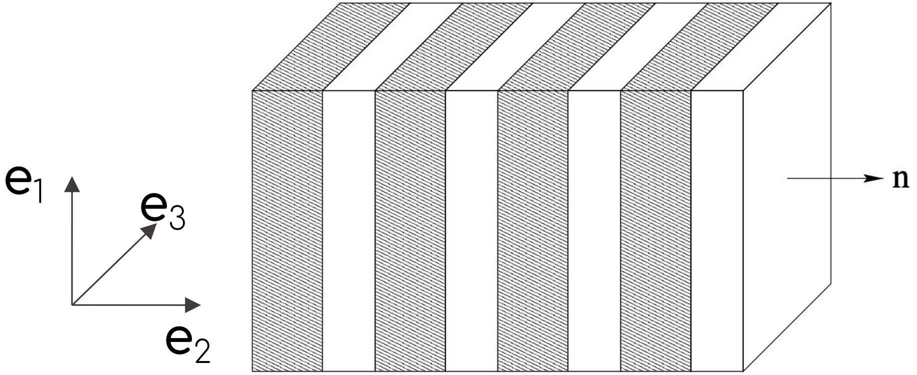
Isotropic layered media: Analytical solution

K. Karapiperis

G. Backus' solution (1962):

Effective stiffness tensor in 2D

$$\mathbb{C}^* = \begin{pmatrix} C_{11}^* & C_{12}^* & 0 \\ C_{12}^* & C_{22}^* & 0 \\ 0 & 0 & C_{44}^* \end{pmatrix}$$



with components

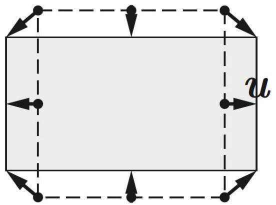
$$C_{11}^* = \left\langle \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \right\rangle + \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^{2}, \quad C_{22}^* = \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1},$$

$$C_{44}^* = \left\langle \frac{1}{\mu} \right\rangle^{-1}, \quad C_{12}^* = \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1},$$

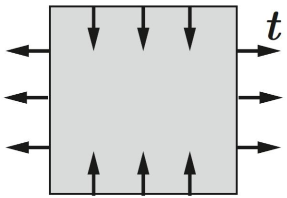
RVE boundary conditions

K.Karapiperis

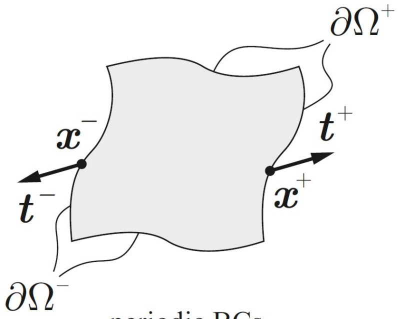
Different types of boundary conditions



affine displacement BCs



uniform traction BCs



periodic BCs

Linearized kin.: $\boldsymbol{u} = \boldsymbol{\varepsilon}_0 \boldsymbol{x}$ $\quad$ $\boldsymbol{t} = \boldsymbol{\sigma}_0 \boldsymbol{n}$ $\quad$ $\boldsymbol{u}^+ = \boldsymbol{u}^- + \boldsymbol{\varepsilon}_0(\boldsymbol{x}^+ - \boldsymbol{x}^-)$, $\boldsymbol{t}^+ = -\boldsymbol{t}^-$

RVE averages: $\langle \boldsymbol{\varepsilon} \rangle = \boldsymbol{\varepsilon}_0$ $\quad$ $\langle \boldsymbol{\sigma} \rangle = \boldsymbol{\sigma}_0$ $\quad$ $\langle \boldsymbol{\varepsilon} \rangle = \boldsymbol{\varepsilon}_0$

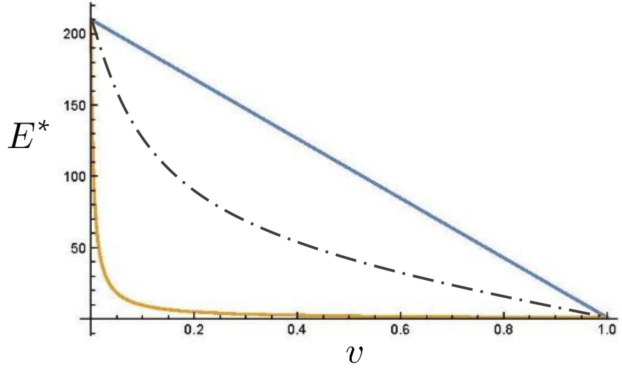
All three satisfy the Hill-Mandel condition

Bounds on effective moduli

Reuss: Lower bound
Voigt: Upper bound

$$\boldsymbol{\varepsilon} \cdot \mathbb{C}^*_{\mathrm{Reuss}} \boldsymbol{\varepsilon} \quad \leq \quad \boldsymbol{\varepsilon} \cdot \mathbb{C}^* \boldsymbol{\varepsilon} \quad \leq \quad \boldsymbol{\varepsilon} \cdot \mathbb{C}^*_{\mathrm{Voigt}} \boldsymbol{\varepsilon}$$

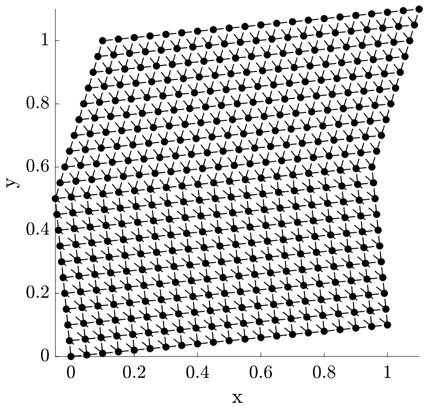
$$\left(\frac{1-v}{E_1} + \frac{v}{E_2} \right)^{-1} \quad \leq \quad E^* \quad \leq \quad (1-v) E_1 + v\, E_2$$



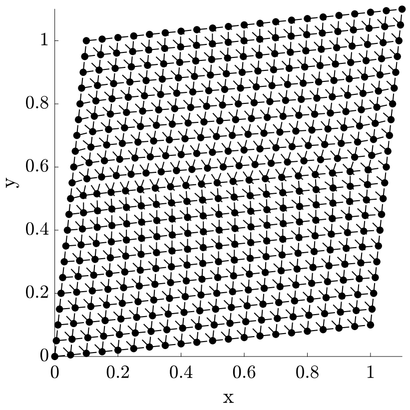
Quiz

Assign the correct boundary conditions to each case.

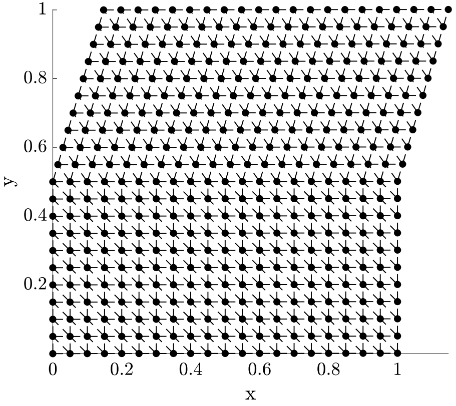
a)



b)



c)

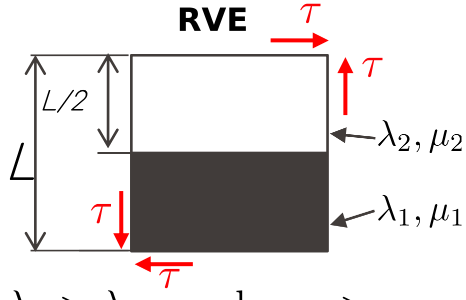


- a)
- b)
- c)

?

- Affine displacement
- Periodic BCs
- Uniform traction

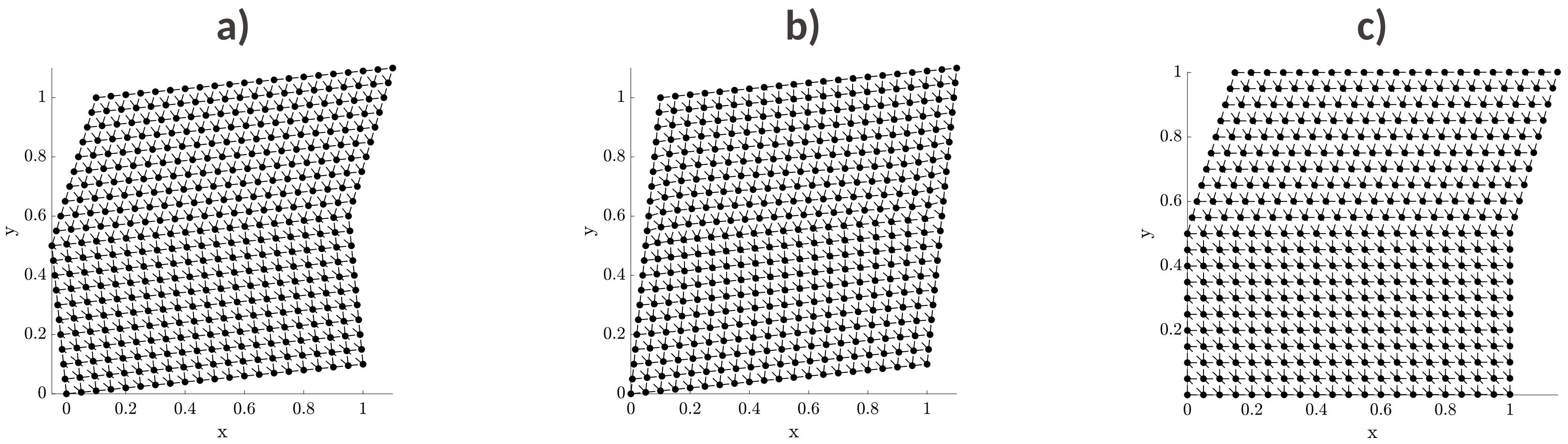
RVE



$\lambda_1 > \lambda_2$ and $\mu_1 > \mu_2$

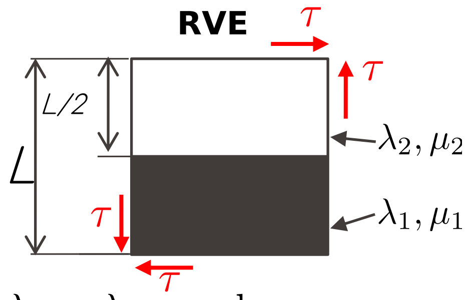
Quiz

Assign the correct boundary conditions to each case.



- a) → Periodic BCs
- b) → Affine displacement
- c) → Uniform traction

Affine displacement
Periodic BCs
Uniform traction

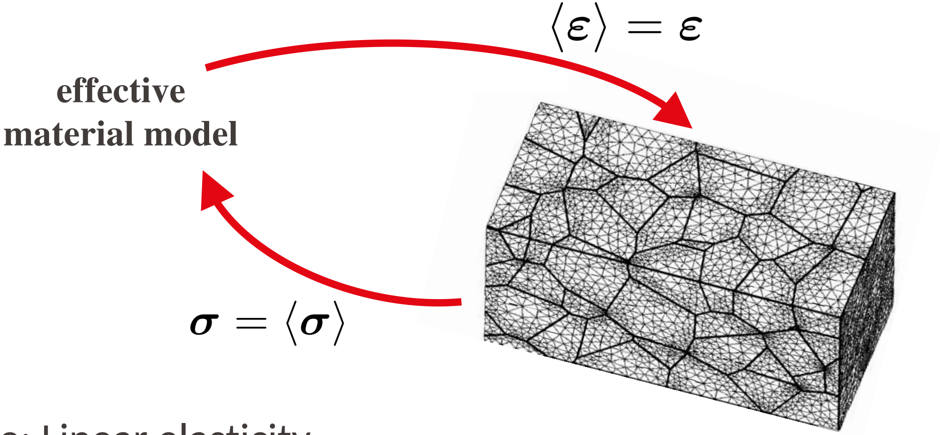


$\lambda_1 > \lambda_2$ and $\mu_1 > \mu_2$

Computational Homogenization

Strategy:

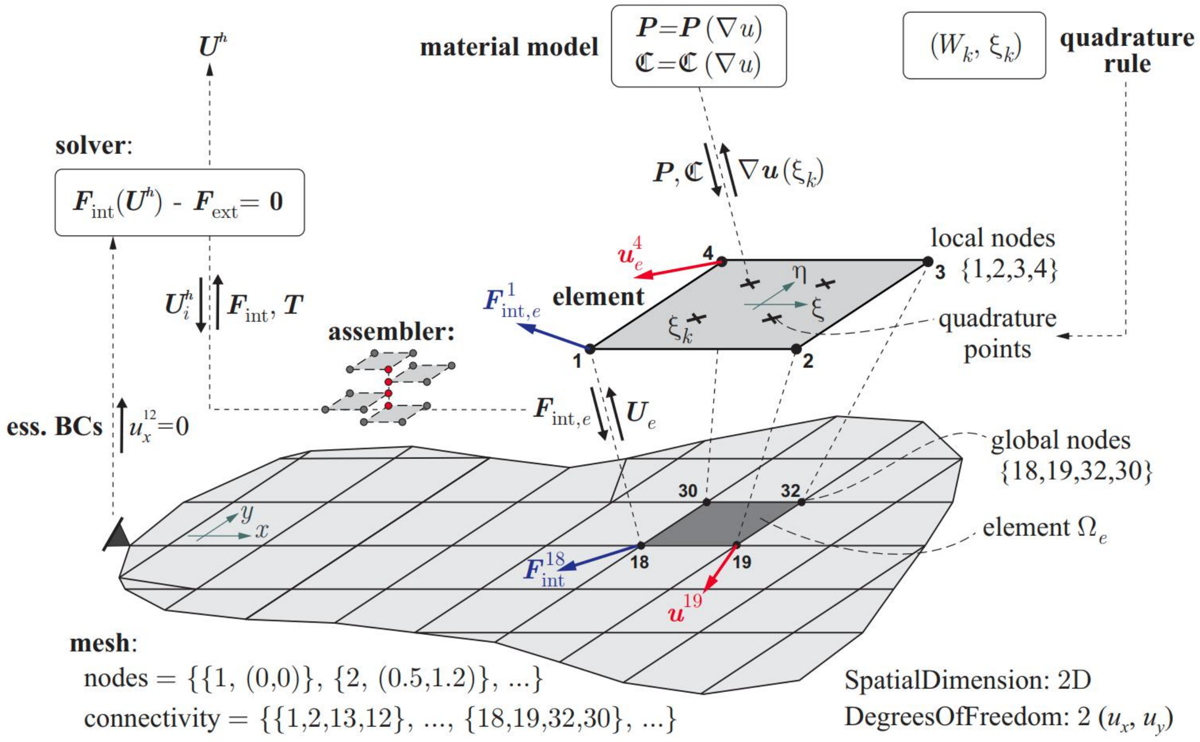
- Compute the effective constitutive response of the material by solving a lower-scale boundary value problem from an RVE:



- Special case: Linear elasticity

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \mathbb{C}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}) \qquad \Rightarrow \qquad \langle \boldsymbol{\sigma} \rangle = \mathbb{C}^* \langle \boldsymbol{\varepsilon} \rangle$$

FEM Overview



Python implementation

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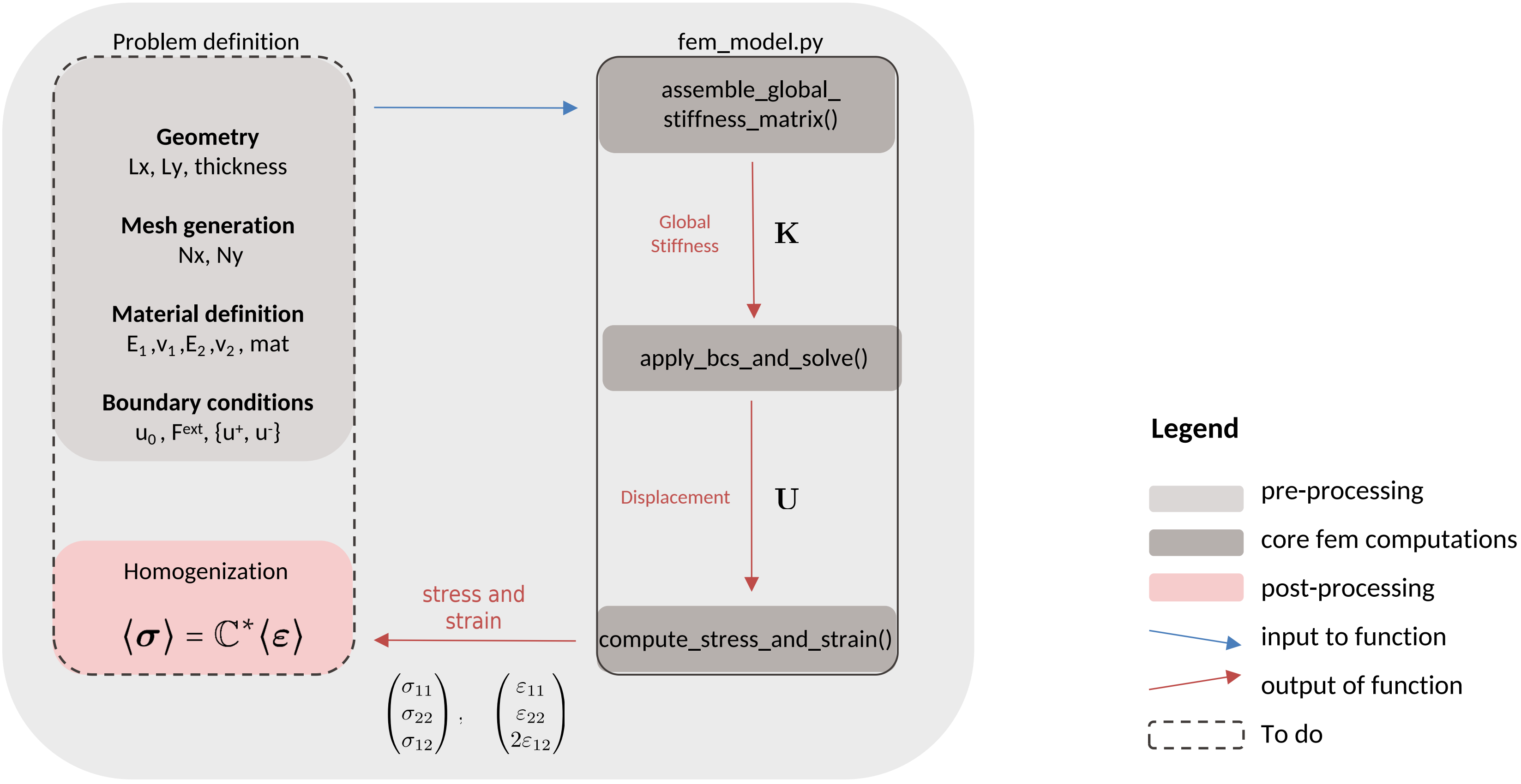
homogenized_stiffness_*.py



Exercise - Laminate

K.Karapiperis

Grid size:

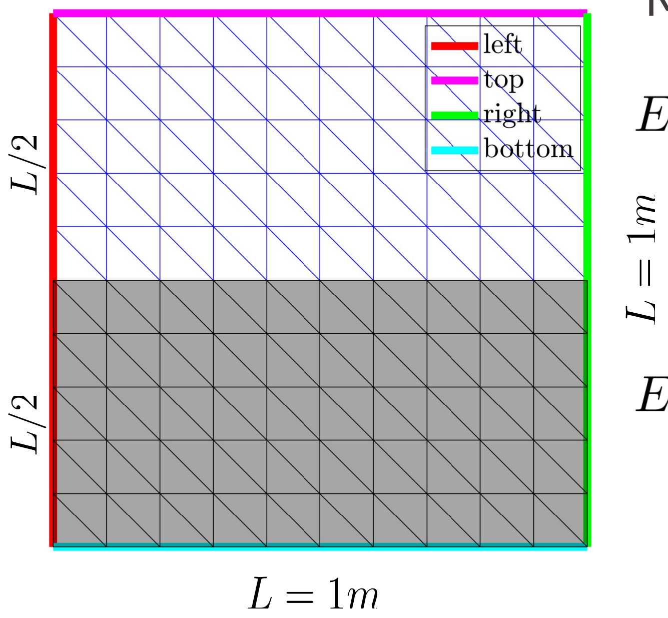
$$N = 11$$

Find:

$$\begin{pmatrix} \mathbb{C}^*_{1111} & \mathbb{C}^*_{1122} & \mathbb{C}^*_{1112} \\ \mathbb{C}^*_{2211} & \mathbb{C}^*_{2222} & \mathbb{C}^*_{2212} \\ \mathbb{C}^*_{1211} & \mathbb{C}^*_{1222} & \mathbb{C}^*_{1212} \end{pmatrix}$$

using:

- Affine displacement BCs
- Uniform traction BCs
- Analytical solution



Material properties:

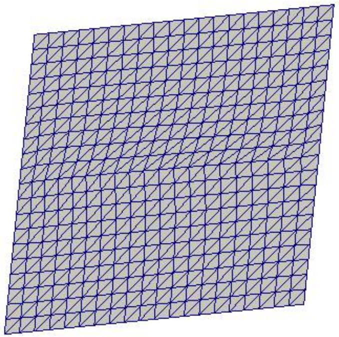
$$E_2 = 1\text{GPa}, \quad \nu_2 = 0.45$$

$$E_1 = 100\text{GPa}, \quad \nu_1 = 0.3$$

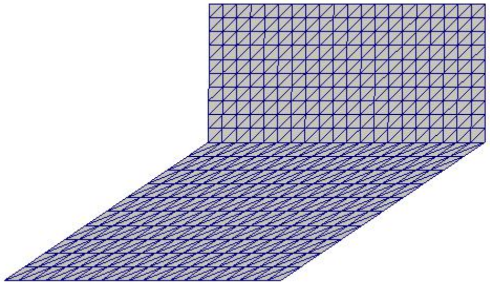
Exercise - Laminate

We will impose two types of boundary conditions:

Affine displacement | **Uniform traction**



$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{\varepsilon}\boldsymbol{x}$$



$$\boldsymbol{t}(\boldsymbol{x}) = \boldsymbol{\sigma}\boldsymbol{n}$$

RVE assumptions:
- two-phase composite
- each phase is homogeneous, isotropic, linear elastic

Extracting elastic moduli

From BCs to effective moduli:

With the computed average stress and strain tensor, we can write

$$\langle \boldsymbol{\sigma} \rangle = \mathbb{C}^* \langle \boldsymbol{\varepsilon} \rangle$$

where we assume that $\mathbb{C}^*$ has **9 unknowns** (due to numerical errors), viz

$$\begin{pmatrix} \langle \sigma_{11} \rangle \\ \langle \sigma_{22} \rangle \\ \langle \sigma_{12} \rangle \end{pmatrix} = \begin{pmatrix} \mathbb{C}^*_{1111} & \mathbb{C}^*_{1122} & \mathbb{C}^*_{1112} \\ \mathbb{C}^*_{2211} & \mathbb{C}^*_{2222} & \mathbb{C}^*_{2212} \\ \mathbb{C}^*_{1211} & \mathbb{C}^*_{1222} & \mathbb{C}^*_{1212} \end{pmatrix} \begin{pmatrix} \langle \varepsilon_{11} \rangle \\ \langle \varepsilon_{22} \rangle \\ 2\langle \varepsilon_{12} \rangle \end{pmatrix}$$

Extracting elastic moduli

K. Karapiperis

9 unknown moduli in 2D:

- apply three (linearly independent strain states)
- compute the three resulting stress states
- solve for the nine unknowns or obtain directly, e.g.,

$$\begin{pmatrix} \langle\sigma_{11}\rangle \\ \langle\sigma_{22}\rangle \\ \langle\sigma_{12}\rangle \end{pmatrix} = \begin{pmatrix} \mathbb{C}^*_{1111} & \mathbb{C}^*_{1122} & \mathbb{C}^*_{1112} \\ \mathbb{C}^*_{2211} & \mathbb{C}^*_{2222} & \mathbb{C}^*_{2212} \\ \mathbb{C}^*_{1211} & \mathbb{C}^*_{1222} & \mathbb{C}^*_{1212} \end{pmatrix} \begin{pmatrix} \langle\varepsilon\rangle \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbb{C}^*_{1111}\langle\varepsilon\rangle \\ \mathbb{C}^*_{2211}\langle\varepsilon\rangle \\ \mathbb{C}^*_{1211}\langle\varepsilon\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle\sigma_{11}\rangle \\ \langle\sigma_{22}\rangle \\ \langle\sigma_{12}\rangle \end{pmatrix} = \begin{pmatrix} \mathbb{C}^*_{1111} & \mathbb{C}^*_{1122} & \mathbb{C}^*_{1112} \\ \mathbb{C}^*_{2211} & \mathbb{C}^*_{2222} & \mathbb{C}^*_{2212} \\ \mathbb{C}^*_{1211} & \mathbb{C}^*_{1222} & \mathbb{C}^*_{1212} \end{pmatrix} \begin{pmatrix} 0 \\ \langle\varepsilon\rangle \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbb{C}^*_{1122}\langle\varepsilon\rangle \\ \mathbb{C}^*_{2222}\langle\varepsilon\rangle \\ \mathbb{C}^*_{1222}\langle\varepsilon\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle\sigma_{11}\rangle \\ \langle\sigma_{22}\rangle \\ \langle\sigma_{12}\rangle \end{pmatrix} = \begin{pmatrix} \mathbb{C}^*_{1111} & \mathbb{C}^*_{1122} & \mathbb{C}^*_{1112} \\ \mathbb{C}^*_{2211} & \mathbb{C}^*_{2222} & \mathbb{C}^*_{2212} \\ \mathbb{C}^*_{1211} & \mathbb{C}^*_{1222} & \mathbb{C}^*_{1212} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \langle\varepsilon\rangle \end{pmatrix} = \begin{pmatrix} \mathbb{C}^*_{1112}\langle\varepsilon\rangle \\ \mathbb{C}^*_{2212}\langle\varepsilon\rangle \\ \mathbb{C}^*_{1212}\langle\varepsilon\rangle \end{pmatrix}$$

Extracting elastic moduli

K.Karapiperis

When applying traction, we obtain the **compliance tensor**:

$$\begin{pmatrix} \langle \varepsilon_{11} \rangle \\ \langle \varepsilon_{22} \rangle \\ 2\langle \varepsilon_{12} \rangle \end{pmatrix} = \begin{pmatrix} \mathbb{S}^*_{1111} & \mathbb{S}^*_{1122} & \mathbb{S}^*_{1112} \\ \mathbb{S}^*_{2211} & \mathbb{S}^*_{2222} & \mathbb{S}^*_{2212} \\ \mathbb{S}^*_{1211} & \mathbb{S}^*_{1222} & \mathbb{S}^*_{1212} \end{pmatrix} \begin{pmatrix} \langle \sigma_{11} \rangle \\ \langle \sigma_{22} \rangle \\ \langle \sigma_{12} \rangle \end{pmatrix}$$

Enforce a uniaxial stress-state, e.g. in the 1-direction

$$\begin{pmatrix} \langle \varepsilon_{11} \rangle \\ \langle \varepsilon_{22} \rangle \\ 2\langle \varepsilon_{12} \rangle \end{pmatrix} = \begin{pmatrix} \mathbb{S}^*_{1111} & \mathbb{S}^*_{1122} & \mathbb{S}^*_{1112} \\ \mathbb{S}^*_{2211} & \mathbb{S}^*_{2222} & \mathbb{S}^*_{2212} \\ \mathbb{S}^*_{1211} & \mathbb{S}^*_{1222} & \mathbb{S}^*_{1212} \end{pmatrix} \begin{pmatrix} \langle \sigma \rangle \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbb{S}^*_{1111} \langle \sigma \rangle \\ \mathbb{S}^*_{2211} \langle \sigma \rangle \\ \mathbb{S}^*_{1211} \langle \sigma \rangle \end{pmatrix}$$

Invert to get the tangent stiffness tensor

$$\begin{pmatrix} \langle \sigma_{11} \rangle \\ \langle \sigma_{22} \rangle \\ \langle \sigma_{12} \rangle \end{pmatrix} = \begin{pmatrix} \mathbb{C}^*_{1111} & \mathbb{C}^*_{1122} & \mathbb{C}^*_{1112} \\ \mathbb{C}^*_{2211} & \mathbb{C}^*_{2222} & \mathbb{C}^*_{2212} \\ \mathbb{C}^*_{1211} & \mathbb{C}^*_{1222} & \mathbb{C}^*_{1212} \end{pmatrix} \begin{pmatrix} \langle \varepsilon_{11} \rangle \\ \langle \varepsilon_{22} \rangle \\ 2\langle \varepsilon_{12} \rangle \end{pmatrix}$$

Exercise - Laminate

Let’s move to the Python notebook

Solution

Solution

Affine disp BCs:

$$\begin{pmatrix} 61.45 & 11.82 & 0.18 \\ 11.82 & 24.67 & 0.42 \\ 0.18 & 0.42 & 14.13 \end{pmatrix} \text{GPa}$$

Uniform traction BCs:

$$\begin{pmatrix} 8.31 & 5.95 & 0.0 \\ 5.95 & 7.34 & 0.0 \\ 0.0 & 0.0 & 0.68 \end{pmatrix} \text{GPa}$$

Analytical:

$$\begin{pmatrix} 58.44 & 4.60 & 0.0 \\ 4.60 & 7.38 & 0.0 \\ 0.0 & 0.0 & 0.68 \end{pmatrix} \text{GPa}$$